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Mixed convection from a sphere at small Reynolds and Grashof numbers

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Consideration is given to the effects of gravity which arise when a heated sphere, maintained at a steady uniform temperature, is located in a vertical uniform stream. Restricting analysis to a medium of unit Prandtl number (σ), the method of matched asymptotic expansions is employed in obtaining solutions for the velocity, temperature and pressure fields in the limit: $G = o(R^2)$, $R \downarrow 0$ (G and R being, respectively, the Grashof and Reynolds numbers). Based on these results, conjectures are formed about the corresponding pure natural convection problem.

1. Preliminaries: the natural convection problem

In the present section, we review the small Grashof number natural convection problem in order to indicate the attendant difficulties and to elicit the similarities with the (analytically simpler) mixed convection problem (§2). Although of interest in its own right, the mixed convection phenomenon may also be used to infer the behaviour of the natural convection flow; such inferences form the basis of §3.

1.1. *The governing equations*

Solutions are sought for the flow field arising from a sphere of radius a and temperature t_w which is located in an unbounded medium in the presence of gravity, g (acting in the negative x-direction). The undisturbed fluid, at rest with respect to the sphere, is taken to be of uniform temperature, t_∞, and density, ρ_∞. It is further assumed that the fluid is of constant transport properties, that viscous dissipation is negligible and that the fluid density is uniform except as it relates to the buoyancy effect (the density then taken to be only a function of temperature). The resulting governing equations are:

$$\nabla . \hat{\mathbf{v}} = 0, \tag{1}$$

$$(\hat{\mathbf{v}} . \nabla)\hat{\mathbf{v}} = -\rho_\infty^{-1}\nabla\hat{p} + \nu\nabla^2\hat{\mathbf{v}} + g\beta\hat{T}\mathbf{i}_x, \tag{2}$$

$$(\hat{\mathbf{v}} . \nabla)\hat{T} = \alpha\nabla^2\hat{T}, \tag{3}$$

where $\hat{\mathbf{v}}$, $\hat{T}$, $\hat{p}$ are, respectively, the fluid velocity, the temperature difference ($\equiv t - t_\infty$) and the pressure arising from the fluid motion, ν the kinematic viscosity, β the coefficient of thermal expansion and α the thermal diffusivity.

The boundary conditions are that $\hat{\mathbf{v}} = 0$, $\hat{T} = t_w - t_\infty$ at $\hat{r} = a$ ($\hat{r}$ being the dimensional radial co-ordinate) and that $\hat{\mathbf{v}}$ be bounded and $\hat{T}$ vanish as $\hat{r} \to \infty$ (more precisely, as $\hat{r} \to \infty$, $\hat{\mathbf{v}}$ should vanish 'almost everywhere', i.e. everywhere except in the narrow wake region above the body, wherein $\hat{\mathbf{v}}$ should be bounded).

1.2. *The inner region*

Employing a regular perturbation scheme for small values of the Grashof number, $G \equiv g\beta(t_w - t_\infty)a^3/\nu^2$, one has:

$$\hat{\mathbf{v}} = \frac{\nu}{a}\mathbf{v}^*(r,\theta;G) \sim \frac{\nu}{a}\sum_{n=1}\delta_n^*(G)\mathbf{v}_n^*(r,\theta), \tag{4}$$

$$\hat{T} = (t_w - t_\infty)T^*(r,\theta;G) \sim (t_w - t_\infty)\sum_{n=0}\phi_n^*(G)T_n^*(r,\theta), \tag{5}$$

$$\hat{p} = \frac{\rho_\infty \nu^2}{a^2}p^*(r,\theta;G) \sim \frac{\rho_\infty \nu^2}{a^2}\sum_{n=1}\delta_n^*(G)p^*(r,\theta), \tag{6}$$

where $r \equiv \hat{r}/a$, θ is the angular co-ordinate measured from the positive x-axis and

$$\lim_{G\to 0}\frac{\delta_{n+1}^*(G)}{\delta_n^*(G)} = 0, \text{ etc.}$$

In non-dimensional form, the governing equations now become:

$$\nabla . \mathbf{v}^* = 0, \tag{7}$$

$$(\mathbf{v}^* . \nabla)\mathbf{v}^* = -\nabla p^* + \nabla^2 \mathbf{v}^* + GT^*\mathbf{i}_x, \tag{8}$$

$$(\mathbf{v}^* . \nabla)T^* = (1/\sigma)\nabla^2 T^*, \tag{9}$$

where $\sigma \equiv \nu/\alpha$, the Prandtl number. The appropriate boundary conditions are: $\mathbf{v}^*(1,\theta;G) = 0$, $T^*(1,\theta;G) = 1$, and $\mathbf{v}^*$ bounded, $T^* \sim 0$ as $r \to \infty$.

If $G = 0$, the problem reduces to that of pure conduction from an isothermal heated sphere and results in

$$T^* = 1/r, \quad \mathbf{v}^* = 0 = p^*.$$

Hence, in expansion (5) above,

$$\phi_0^*(G)T_0^*(r,\theta) = 1/r. \tag{10}$$

From (5) and (10) we see that T^* is $O(1)$, indicating that the buoyancy force in (8) is $O(G)$. It is therefore natural to set $\delta_1^*(G) = G$, resulting in

$$\nabla^2 \mathbf{v}_1^* - \nabla p_1^* = -(1/r)\mathbf{i}_x. \tag{11}$$

Expressed in terms of the stream function, $\psi_1^*(r,\theta)$, where

$$\mathbf{v}_1^* . \mathbf{i}_r = \frac{1}{r^2 \sin\theta}\frac{\partial \psi_1^*}{\partial \theta}, \quad \mathbf{v}_1^* . \mathbf{i}_\theta = -\frac{1}{r\sin\theta}\frac{\partial \psi_1^*}{\partial r},$$

the curl of (11) becomes
$$D^2\psi_1^* = \frac{\sin^2\theta}{r}, \tag{12}$$

where
$$D \equiv \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right).$$

Subject to the conditions that ψ_1^* be zero along $\theta = 0, \pi$ and have a double zero at $r = 1$, the general solution to (12) is

$$\psi_1^* = -\tfrac{1}{8}r^3\sin^2\theta + \sum_1^\infty \{A_n[r^{n+1} - (n+\tfrac{1}{2})r^{2-n} + (n-\tfrac{1}{2})r^{-n}] + B_n[r^{n+3} - (n+\tfrac{3}{2})r^{2-n} + (n+\tfrac{1}{2})r^{-n}]\}G_n(\cos\theta), \quad (13)$$

where G_n is the Gegenbauer polynomial defined by

$$G_n(z) \equiv \int_{-1}^{z} P_n(t)\,\mathrm{d}t,$$

P_n being the Legendre polynomial of degree n. Applying the principle of minimum singularity (cf. Van Dyke 1964), it follows that only A_1 and A_2 may be non-zero.

One notes that, as $r \to \infty$, the $O(r^3)$ behaviour of ψ_1^* corresponds to an $O(r)$ behaviour in $\mathbf{v}_1^*$, thus precluding satisfaction of the boundary condition at infinity. This clearly indicates the inadequacy of the regular perturbation expansion procedure. As in the corresponding small Reynolds number problem, the convection effect must be considered in the distant region. That is, there exists an outer region in which the convective, diffusive and buoyant effects are of the same order of magnitude.

1.3. *The outer region*

Denoting the characteristic length and speed in the outer region by L and U_B, the requirement that the convective and diffusive effects be of the same order results in $L = \nu/U_B$. In order to determine U_B, one notes from (4) and the $O(r)$ behaviour of $\mathbf{v}_1^*$ (for large r) that, in the outer region,

$$U_B \sim |\hat{\mathbf{v}}| = \frac{\nu}{a}G.O(r) \sim \frac{\nu}{a}G\frac{L}{a} = \frac{\nu}{a}G\frac{\nu}{U_B a},$$

that is,
$$U_B = (\nu/a)G^{\frac{1}{2}} = \sqrt{(g\beta(t_w - t_\infty)a)}. \quad (14)$$

(Alternatively, this result is obtainable by equating the inertial and buoyant forces, noting that $\hat{T} \sim (t_w - t_\infty).O(a/L)$ in the outer region.) Hence, with $\rho^* \equiv \hat{r}/L = rG^{\frac{1}{2}}$, the appropriate asymptotic expansions in the outer region are:

$$\hat{\mathbf{v}} = \sqrt{(g\beta(t_w - t_\infty)a)}\mathbf{V}^*(\rho^*, \theta; G) \sim \sqrt{(g\beta(t_w - t_\infty)a)} \sum_{n=1} \Delta_n^*(G)\mathbf{V}_n^*(\rho^*, \theta), \quad (15)$$

$$\hat{T} = (t_w - t_\infty)\mathscr{T}^*(\rho^*, \theta; G) \sim (t_w - t_\infty) \sum_{n=1} \Phi_n^*(G)\mathscr{T}_n^*(\rho^*, \theta), \quad (16)$$

$$\hat{p} = \rho_\infty g\beta(t_w - t_\infty)a\mathscr{P}^*(\rho^*, \theta; G) \sim \rho_\infty g\beta(t_w - t_\infty)a \sum_{n=1} \Delta_n^*(G)\mathscr{P}_n^*(\rho^*, \theta), \quad (17)$$

where, based upon the above considerations, $\Delta_1^*(G) = O(1)$ and $\Phi_1^*(G) = O(G^{\frac{1}{2}})$. In particular, setting $\Delta_1^*(G) = 1$ and $\Phi_1^*(G) = G^{\frac{1}{2}}$, the governing equations for $\mathbf{V}_1^*, \mathscr{T}_1^*, \mathscr{P}_1^*$ become:

$$\nabla.\mathbf{V}_1^* = 0, \quad (18)$$

$$(\mathbf{V}_1^*.\nabla)\mathbf{V}_1^* = -\nabla\mathscr{P}_1^* + \nabla^2\mathbf{V}_1^* + \mathscr{T}_1^*\mathbf{i}_x, \quad (19)$$

$$(\mathbf{V}_1^*.\nabla)\mathscr{T}_1^* = (1/\sigma)\nabla^2\mathscr{T}_1^*. \quad (20)$$

The boundary conditions are that $\mathbf{V}_1^*$ be bounded and $\mathscr{T}_1^*$ vanish as $\rho \to \infty$; matching considerations with respect to T_0^* and ψ_1^* require that $\mathscr{T}_1^* \sim 1/\rho^*$ as $\rho^* \to 0$ and that the $O(\rho^{*3})$ behaviour of ψ_1^* include the term $-\frac{1}{8}\rho^{*3}\sin^2\theta$.

It is at this point that the small Grashof number problem distinguishes itself from the small Reynolds number forced convection problem. In the latter, the momentum and energy equations in the outer region are uncoupled and linear, the convective operator being linearized about the uniform stream velocity; in the former, as evidenced by (18)–(20), the momentum and energy equations are coupled and non-linear in the outer region.

Although exact solutions to (18)–(20) are not forthcoming, the behaviour of such solutions for ρ^* large can be obtained by means of co-ordinate perturbation expansions. Such expansions usually regard the sphere as a point source of heat, Q, and are not limited to small G, requiring rather that $\hat{r} \gg \nu/U^*$, U^* being the characteristic speed. Employing dimensional and energy conservation considerations, one readily finds that $U^* = \sqrt{(g\beta Q/k)}$, k being the thermal conductivity of the fluid. For small G, however, Q is essentially due to pure conduction, i.e. $Q \approx 4\pi ak(t_w - t_\infty)$, indicating that then the characteristic speed is, indeed, $\sqrt{(g\beta(t_w - t_\infty)a)}$.

In terms of the present problem, the well-known co-ordinate perturbation solutions (cf. Yih 1953) indicate a paraboloidal wake region: $\theta = O(\rho^{*-\frac{1}{2}})$ as $\rho^* \to \infty$, in which $\mathbf{V}_1^* = O(1)$, $\mathscr{T}_1^* = O(1/\rho^*)$. Towards the edge of the wake, $\mathbf{V}_1^* \sim -A(\sigma)\mathbf{i}_{\tilde{\omega}^*}/\tilde{\omega}^*$($\tilde{\omega}^*$ being the cylindrical radial co-ordinate, $\rho^*\sin\theta$, and $A(\sigma)$ a positive constant (for given σ)) with the vorticity and temperature vanishing algebraically. In particular, for the case $\sigma = 1$, Yih (1953) obtained a closed-form solution for the wake which, in terms of the present analysis, is expressible as

$$\mathbf{V}_1^* \sim \frac{\sqrt{2}}{(1+\eta^{*2})^2}\mathbf{i}_x, \quad (\eta^* \text{fixed}, \rho^* \to \infty), \tag{21}$$

$$\mathscr{T}_1^* \sim \frac{1}{\rho^*}\frac{\frac{4}{3}}{(1+\eta^{*2})^3}, \quad (\eta^* \text{ fixed}, \rho^* \to \infty), \tag{22}$$

where
$$\eta^{*2} = \frac{1}{6\sqrt{2}}\frac{\tilde{\omega}^{*2}}{x^*}.$$

1.4. *Previous analyses*

The classic paper for the small Grashof number problem is that of Mahony (1957). Considering both the sphere and circular cylinder cases, Mahony notes the futility of obtaining exact solutions for the outer region and, instead, seeks similarity solutions to (18)–(20) (and to the corresponding equations for the circular cylinder case) by assuming the existence of a vertical plume (wake) in this region. In fact, the wake region corresponds to a co-ordinate perturbation and is only valid for $\rho^* \gg 1$ (i.e. $\hat{r} \gg \nu/U_B = aG^{-\frac{1}{2}}$); hence, as noted by Mahony, it is impossible to match the wake solutions with the regular perturbation expansions of the inner region. However, for the circular cylinder case, Mahony does patch the temperature of the wake with that of the inner region at a particular point along the positive x-axis, obtaining the qualitative result that, in the inner region,

$$\hat{T} \sim (t_w - t_\infty)[1 + \lambda(G)\log r] \quad (r \text{ fixed}, G \to 0), \tag{23}$$

where $\lambda(G) = O(1/\log G)$. This result is completely analogous to that obtained in the corresponding small Reynolds number forced convection problem, the inner temperature expansion then being (cf. Kassoy 1967 or Hieber & Gebhart 1968):

$$\hat{T} \sim (t_w - t_\infty)\left[1 + \frac{\log r}{\log(\gamma\sigma R/4)}\right] \quad (r \text{ fixed}, R \to 0), \tag{24}$$

where $R \equiv aU_\infty/\nu$, the Reynolds number (U_∞ being the magnitude of the uniform stream). Although (23) and (24) are based upon pure conduction in the inner region (as is evidenced by '$\log r$'), the G, R dependent coefficients in (23), (24) indicate that the inner temperature distribution is still dependent upon thermal convection; this dependence arises from matching considerations, indicating that the inner thermal field is 'induced' by the velocity field in the outer region (the inner temperature distribution being uniform when G, $R = 0$). This similar structure between the natural and forced convection from a circular cylinder leads one to suspect an analogous similarity for the sphere case. (Although it is realized that the circular cylinder case is probably of more practical (and even theoretical) interest, the present analysis is primarily concerned with the sphere since, as indicated below, the expansion technique employed in §2 is non-applicable to the circular cylinder case.)

Recently, Fendell (1968) has treated the small Grashof number problem for the sphere, approximating the equations in the outer region with Oseen's equation. The magnitude of the hypothetical uniform stream is based upon the co-ordinate perturbation solution, the latter indicating a constant velocity along the positive x-axis which is proportional to $\sqrt{(g\beta(t_w - t_\infty)a)}$, the constant of proportionality being a function of the Prandtl number (which, as seen from (15) and (21), is $\sqrt{2}$ when $\sigma = 1$). At best, such a procedure can be expected to yield qualitative information. In particular, Fendell finds that the velocity in the inner region is of order $(\nu/a)G^{\frac{1}{2}}$ rather than $(\nu/a)G$.

2. The mixed convection problem

In the present section, the physical situation differs from that of §1 in the one respect that the undisturbed fluid is now moving at speed U_∞ with respect to the sphere. For simplicity, the direction of this uniform stream is taken to be along the positive x-axis (opposite to the direction of gravity); in addition, the Prandtl number is assumed to be unity.

The basic assumption of the current theory is that Oseen linearization about U_∞ is valid in the outer region; this requires that the Reynolds number be small and that the gravity-induced velocity in the outer region be small with respect to U_∞. Since (as is found below) Oseen linearization leads to a gravity-induced velocity of order $g\beta(t_w - t_\infty)a/U_\infty$, it follows that the present analysis is restricted to $U_B \ll U_\infty$ with $R \ll 1$ or, more precisely, $G = o(R^2)$ as $R \to 0$. In effect, then, the current theory is for the case in which the small Reynolds number forced convection flow is perturbed by gravity effects.

One notes that implicit in the above argument is the assumption that the resulting gravity-induced velocity is bounded as $\hat{r}\rightarrow\infty$. That this is indeed the case can be readily verified as follows. Applying the Oseen linearization to the energy equation in the outer region (and noting that the leading term in this region must match the pure conduction temperature of the inner region), it follows that, to first approximation, the temperature distribution in the outer region is independent of gravity and is, indeed, merely that of the forced flow case. If one substitutes this known temperature distribution (equation (29) below) into the buoyancy term of the momentum equation and integrates the resulting force over a spherical volume of radius Γ, one finds the total buoyant force to be $O(\Gamma)$ for large Γ; assuming this force is evidenced as a momentum flux in the paraboloidal wake, it follows that the gravity-induced velocity in the wake is $O(1)$ as $\Gamma\rightarrow\infty$. (A similar calculation for the circular cylinder case results in a velocity of $O(\Gamma^{\frac{1}{2}})$ as $\Gamma\rightarrow\infty$, indicating that the Oseen linearization breaks down far out in the wake, thus precluding application of the present theory to this case.)

2.1. *Outer region*

In the outer (Oseen) region the characteristic length is now ν/U_∞ and the appropriate non-dimensionalization is:

$$\hat{\mathbf{v}} = U_\infty \mathbf{V}(\rho,\theta;R;\epsilon),$$
$$\hat{T} = (t_w-t_\infty)\mathscr{T}(\rho,\theta;R;\epsilon),$$
$$\hat{p} = \rho_\infty U_\infty^2 \mathscr{P}(\rho,\theta;R;\epsilon),$$

resulting in the equations (unit Prandtl number):

$$\nabla . \mathbf{V} = 0, \tag{25}$$

$$(\mathbf{V}.\nabla)\mathbf{V} = -\nabla\mathscr{P}+\nabla^2\mathbf{V}+(G/R^3)\mathscr{T}\mathbf{i}_x, \tag{26}$$

$$(\mathbf{V}.\nabla)\mathscr{T} = \nabla^2\mathscr{T}, \tag{27}$$

where $\rho\equiv\hat{r}/(\nu/U_\infty)$ and $\epsilon=\epsilon(R;G)$, a function yet to be determined. The boundary conditions are that $\mathbf{V}$ be bounded and $\mathscr{T}$ vanish as $\rho\rightarrow\infty$; additional conditions result from matching considerations in the region $\rho\rightarrow 0$.

It is convenient to employ superscripts 'F' and 'B' to denote those quantities which arise solely from forced convection and those which depend upon the buoyancy effect. For example,

$$\mathbf{V}(\rho,\theta;R;\epsilon) = \mathbf{V}^F(\rho,\theta;R)+\mathbf{V}^B(\rho,\theta;R;\epsilon).$$

Hence the expansion for $\mathbf{V}^F$ corresponds to the limit: ρ fixed, $G\equiv 0$, $R\rightarrow 0$, the well-known result being (cf. Kaplun & Lagerstrom 1957 or Proudman & Pearson 1957):

$$\mathbf{V}^F(\rho,\theta;R) \sim \mathbf{i}_x+R\mathbf{V}_1^F(\rho,\theta)+O(R^2),$$

where $\mathbf{V}_1^F$ is based upon the stream function (non-dimensionalized with respect to ν^2/U_∞):

$$\Psi_1^F(\rho,\theta) = -\tfrac{3}{2}(1+\cos\theta)\,(1-e^{-\frac{1}{2}\rho(1-\cos\theta)}).$$

In the same limit (cf. Acrivos & Taylor 1962 or Rimmer 1968),

$$\mathscr{T}^F(\rho,\theta;R) \sim R\mathscr{T}_1^F(\rho,\theta)+R^2\mathscr{T}_2^F(\rho,\theta)+\ldots, \tag{28}$$

where

$$\mathscr{T}_1^F(\rho,\theta) = (1/\rho)e^{-\frac{1}{2}\rho(1-\cos\theta)} \tag{29}$$

and $\mathscr{T}_2^F(\rho,\theta)$ is expressible in terms of an infinite series (cf. appendix C).

Invoking the basic assumption that $\mathbf{V}^B$ is everywhere small with respect to $\mathbf{i}_x$, it follows immediately from (27) that the leading temperature term in the Oseen region is independent of gravity and is, indeed, $R\mathscr{T}_1^F$ (this being the integral which matches the pure conduction temperature distribution of the inner region). Substituting $R\mathscr{T}_1^F$ for $\mathscr{T}$ in (26) indicates that the leading gravity-induced velocity is $O(G/R^2)$. Letting $\epsilon \equiv G/R^2$, it follows that the present analysis is based on the limit: $\epsilon = o(1)$, $R\to 0$ (cf. appendix A for a schematic diagram of the structure of the resulting expansions). Hence, with

$$\left.\begin{aligned}\mathbf{V}^B(\rho,\theta;R;\epsilon) &\sim \sum_{n=1}\Delta_n(R;\epsilon)\mathbf{V}_n^B(\rho,\theta)\\ \mathscr{P}^B(\rho,\theta;R;\epsilon) &\sim \sum_{n=1}\Delta_n(R;\epsilon)\mathscr{P}_n^B(\rho,\theta)\end{aligned}\right\}\quad (\rho\,\text{fixed}, \epsilon = o(1), R\to 0), \tag{30}$$

it follows from the above that $\Delta_1(R;\epsilon) = \epsilon$. Collecting the terms of $O(\epsilon)$ in (26) results in

$$([\partial/\partial x]-\nabla^2)\mathbf{V}_1^B+\nabla\mathscr{P}_1^B = \mathscr{T}_1^F\,\mathbf{i}_x. \tag{31}$$

The corresponding vorticity equation is

$$\left(\mathscr{D}-\cos\theta\frac{\partial}{\partial\rho}+\frac{\sin\theta}{\rho}\frac{\partial}{\partial\theta}\right)\mathscr{D}\Psi_1^B = \left(\frac{1}{\rho}+\tfrac{1}{2}\right)\sin^2\theta\, e^{-\frac{1}{2}\rho(1-\cos\theta)}, \tag{32}$$

where

$$\mathscr{D} \equiv \frac{\partial^2}{\partial\rho^2}+\frac{\sin\theta}{\rho^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right).$$

A particular solution of (32) is

$$\Psi_1^B(\rho,\theta) = \rho(1+\cos\theta)(1-e^{-\frac{1}{2}\rho(1-\cos\theta)}), \tag{33}$$

the irrotational term, $\rho(1+\cos\theta)$, being required in order that Ψ_1^B vanish at $\theta = 0$ in addition to $\theta = \pi$, assuring zero net mass flux through any surface enclosing the body. (Any complementary integral having the proper symmetry and corresponding to a bounded velocity at infinity can be shown to be unmatchable with respect to the inner region.) The irrotational flow in (33) is particularly interesting, the radial and angular velocities being, respectively,

$$\mathbf{V}_{1i}^B.\mathbf{i}_\rho = -\frac{1}{\rho}, \tag{34}$$

$$\mathbf{V}_{1i}^B.\mathbf{i}_\theta = -\frac{1+\cos\theta}{\rho\sin\theta}, \tag{35}$$

(subscript 'i' indicates irrotational flow). It is seen that the angular velocity is singular along $\theta = 0$, the positive x-axis behaving as a line sink of strength 4π per unit length. Hence, the radial inflow arising from (34) is absorbed along the positive x-axis.

In considering the behaviour of $\mathbf{V}_1^B$ for large ρ, it is evident from (33) that the rotational velocity is exponentially small except for the narrow paraboloidal (wake) region: $\theta = O(\rho^{-\frac{1}{2}})$, $\rho \to \infty$. Applying this limit to (34), (35) results in

$$\mathbf{V}_{1i}^B \sim -\frac{2}{\rho\theta}\mathbf{i}_\theta + O(\rho^{-1}) \sim -\frac{2}{\tilde{\omega}}\mathbf{i}_{\tilde{\omega}} + O(\rho^{-1}) \tag{36}$$

($\tilde{\omega}$ being the cylindrical radial co-ordinate, $\rho\sin\theta$), indicating that, with respect to the irrotational flow field, the wake is a line sink. In the same limit, the behaviour of the entire velocity is

$$\mathbf{V}_1^B \sim e^{-\frac{1}{4}\rho\theta^2}\mathbf{i}_\rho + O(\rho^{-\frac{1}{2}}) \sim e^{-\eta^2}\mathbf{i}_x + O(\rho^{-\frac{1}{2}}), \tag{37}$$

where $\eta \equiv \rho^{\frac{1}{2}}\sin\frac{1}{2}\theta \sim \frac{1}{2}\rho^{\frac{1}{2}}\theta$. Hence, as previously indicated, the buoyancy effect induces a velocity of $O(1)$ in the wake region. This flow, (37), represents a mass flux of $O(\rho)$ in the paraboloidal wake, thus necessitating the inflow, (36), from the irrotational region.

For matching purposes, one expands (33) as $\rho \to 0$, obtaining

$$\Psi_1^B(\rho,\theta) \sim \tfrac{1}{2}\rho^2\sin^2\theta - \tfrac{1}{8}\rho^3\sin^2\theta(1-\cos\theta) + O(\rho^4). \tag{38}$$

Therefore,
$$\mathbf{V}_1^B \sim \mathbf{i}_x + O(\rho) \quad (\rho \to 0),$$

indicating that, with respect to the inner region, $\mathbf{V}_1^B$ is a uniform stream in the positive x-direction. (Note: if $t_w < t_\infty$, then $\epsilon < 0$ and the direction of $\epsilon\mathbf{V}_1^B$ is reversed.)

Taking the divergence of (31) results in

$$\nabla^2\mathscr{P}_1^B = \frac{\partial}{\partial x}\mathscr{T}_1^F. \tag{39}$$

The particular solution of (39) which also satisfies (31) is

$$\mathscr{P}_1^B(\rho,\theta) = -\frac{1}{\rho}(1 - e^{-\frac{1}{2}\rho(1-\cos\theta)}), \tag{40}$$

the first term in (40) being required to balance the term $(\partial/\partial x)\mathbf{V}_{1i}^B$ in (31).

2.2. *Inner region*

In the inner (Stokes) region the appropriate non-dimensionalization is

$$\hat{\mathbf{v}} = U_\infty \mathbf{v}(r,\theta;R;\epsilon),$$

$$\hat{T} = (t_w - t_\infty)\,T(r,\theta;R;\epsilon),$$

$$\hat{p} = \frac{\rho_\infty \nu U_\infty}{a}p(r,\theta;R;\epsilon),$$

resulting in the equations:
$$\nabla.\mathbf{v} = 0,$$

$$R(\mathbf{v}.\nabla)\mathbf{v} = -\nabla p + \nabla^2\mathbf{v} + (G/R)T\mathbf{i}_x, \tag{41}$$

$$R(\mathbf{v}.\nabla)T = \nabla^2 T. \tag{42}$$

The boundary conditions are that $\mathbf{v}(1,\theta;R;\epsilon) = 0$ and $T(1,\theta;R;\epsilon) = 1$. Additional conditions arise from matching considerations in the region $r \to \infty$.

With regard to T, it is convenient to use the notation:

$$T(r,\theta;R;\epsilon) = T_0(r,\theta)+T^F(r,\theta;R)+T^B(r,\theta;R;\epsilon),$$

where T_0 is the pure conduction temperature:

$$T_0(r,\theta) = 1/r.$$

The well-known inner expansions for the forced convection flow (corresponding to r fixed, $G \equiv 0$, $R \to 0$) are

$$\psi^F(r,\theta;R) \sim \psi_0^F(r,\theta)+O(R),$$

$$T^F(r,\theta;R) \sim RT_1^F(r,\theta)+O(R^2\log R),$$

where the stream function has been non-dimensionalized with respect to a^2U_∞ and where

$$\psi_0^F(r,\theta) = (\tfrac{1}{2}r^2-\tfrac{3}{4}r+\tfrac{1}{4}r^{-1})\sin^2\theta,$$

$$T_1^F(r,\theta) = \tfrac{1}{2}(r^{-1}-1)+(\tfrac{1}{2}-\tfrac{3}{4}r^{-1}+\tfrac{3}{8}r^{-2}-\tfrac{1}{8}r^{-3})\cos\theta.$$

Since T^B must evidently vanish in the limit r fixed, $\epsilon = o(1)$, $R \to 0$, it follows that the buoyancy force in the inner region is, to first approximation, due to T_0, an effect of $O(G/R)$, i.e. $O(\epsilon R)$. However, matching considerations based upon (30) and (38) indicate the presence of a velocity term of $O(\epsilon)$ in the inner region. Hence, with

$$\psi^B(r,\theta;R;\epsilon) \sim \sum_{n=1} \delta_n(R;\epsilon)\psi_n^B(r,\theta) \quad (r \text{ fixed}, \epsilon = o(1), R \to 0),$$

it follows that $\delta_1(R,\epsilon) = \epsilon$. For convenience, one then sets $\delta_2(R;\epsilon) = \epsilon R$. (It will be found that terms of $O(\epsilon^2)$ are also present in the velocity field. Since the limiting process requires only that ϵ vanish with R, it follows that the relative order of magnitude of ϵR to ϵ^2 is indeterminate; hence, one could just as readily choose $\delta_2(R;\epsilon) = \epsilon^2$.) Matching considerations based upon (38) then indicate that, as $r \to \infty$,

$$\psi_1^B(r,\theta) \sim \tfrac{1}{2}r^2\sin^2\theta, \tag{43}$$

$$\psi_2^B(r,\theta) \sim -\tfrac{1}{8}r^3\sin^2\theta(1-\cos\theta). \tag{44}$$

The governing equation for ψ_1^B is

$$D^2\psi_1^B = 0. \tag{45}$$

From (45), (43) and the conditions that ψ_1^B vanish at $\theta = 0,\pi$ and have a double zero at $r = 1$, it is seen that the problem for ψ_1^B is identical to that for the leading term in ψ^F; hence,

$$\psi_1^B(r,\theta) = (\tfrac{1}{2}r^2-\tfrac{3}{4}r+\tfrac{1}{4}r^{-1})\sin^2\theta. \tag{46}$$

The governing equation for ψ_2^B is

$$D^2\psi_2^B = (-9r^{-2}+\tfrac{27}{2}r^{-3}-\tfrac{9}{2}r^{-5})\sin^2\theta\cos\theta+(\sin^2\theta/r), \tag{47}$$

the first inhomogeneous term arising from the non-linear effect between ψ_0^F and ψ_1^B, the second arising from T_0. A particular integral of (47) is

$$-\tfrac{1}{8}r^3\sin^2\theta+(-\tfrac{3}{8}r^2+\tfrac{9}{16}r+\tfrac{3}{16}r^{-1})\sin^2\theta\cos\theta. \tag{48}$$

The first term in (48) arises from the buoyancy term in (47) and is seen to match the leading term in (44), the second term in (44) corresponding to the complementary integral of (47). From (44) and the conditions that ψ_2^B vanish at $\theta = 0, \pi$ and have a double zero at $r = 1$, it follows that

$$\psi_2^B(r,\theta) = [-\tfrac{1}{8}r^3+\tfrac{1}{4}r-\tfrac{1}{8}r^{-1}+\lambda_2(\tfrac{1}{2}r^2-\tfrac{3}{4}r+\tfrac{1}{4}r^{-1})]\sin^2\theta + (\tfrac{1}{8}r^3-\tfrac{3}{8}r^2+\tfrac{9}{16}r-\tfrac{1}{2}+\tfrac{3}{16}r^{-1})\sin^2\theta\cos\theta. \quad (49)$$

The value (λ_2) of the uniform stream in (49) remains to be determined *via* matching considerations with the velocity of $O(\epsilon R)$ in the outer region. It is readily found that the pressures associated with (46) and (49) are, respectively,

$$p_1^B(r,\theta) = -\tfrac{3}{2}r^{-2}P_1(\cos\theta), \quad (50)$$

$$p_2^B(r,\theta) = (-\tfrac{1}{2}-\tfrac{3}{8}r^{-2}+\tfrac{3}{8}r^{-4}-\tfrac{1}{8}r^{-6})P_0(\cos\theta) + [\tfrac{1}{2}+(\tfrac{1}{2}-\tfrac{3}{2}\lambda_2)r^{-2}]P_1(\cos\theta)+(\tfrac{3}{2}r^{-2}-3r^{-3}+\tfrac{3}{2}r^{-4}-\tfrac{1}{8}r^{-6})P_2(\cos\theta), \quad (51)$$

the constant term in (51), $-\frac{1}{2}P_0(\cos\theta)$, arising from matching requirements with $\mathscr{P}_1^B(\rho,\theta)$.

2.3. *Thermal field* (*outer region*)

From (28) and (30) it is seen that the leading thermal convection effect arising from buoyancy is $(\mathbf{V}_1^B.\nabla)\mathscr{T}_1^F$, an effect of $O(\epsilon R)$. Hence, with

$$\mathscr{T}^B(\rho,\theta;R;\epsilon) \sim \sum_{n=1}\Phi_n(R;\epsilon)\mathscr{T}_n^B(\rho,\theta) \quad (\rho \text{ fixed}, \epsilon = o(1), R\to 0), \quad (52)$$

it follows that $\Phi_1(R,\epsilon) = \epsilon R$, the governing equation for $\mathscr{T}_1^B$ being obtained from the terms of $O(\epsilon R)$ in (27):

$$(\nabla^2-\partial/\partial x)\mathscr{T}_1^B = [\rho^{-3}+\rho^{-2}]\,e^{-\frac{1}{2}\rho(1-\cos\theta)}-[\rho^{-3}+\tfrac{1}{2}(3+\cos\theta)\rho^{-2}]\,e^{-\rho(1-\cos\theta)}. \quad (53)$$

Introducing the transformation

$$\mathscr{T}_1^B(\rho,\theta) = q(\rho,\theta)\,e^{\frac{1}{2}\rho\cos\theta},$$

results in

$$(\nabla^2-\tfrac{1}{4})q(\rho,\theta) = f(\rho,\theta), \quad (54)$$

where

$$f(\rho,\theta) = [\rho^{-3}+\rho^{-2}]\,e^{-\rho/2}-[\rho^{-3}+\tfrac{1}{2}(3+\cos\theta)\rho^{-2}]\,e^{-\rho+\frac{1}{2}\rho\cos\theta}. \quad (55)$$

The Green's function corresponding to the operator in (54) is

$$G(\boldsymbol{\rho},\boldsymbol{\rho}_1) = -\frac{1}{4\pi}\frac{e^{-\frac{1}{2}|\boldsymbol{\rho}-\boldsymbol{\rho}_1|}}{|\boldsymbol{\rho}-\boldsymbol{\rho}_1|},$$

and is expressible as

$$G(\boldsymbol{\rho},\boldsymbol{\rho}_1) = -\frac{1}{4\pi^2}\sum_{n=0}^{\infty}(2n+1)\left[\sqrt{\left(\frac{\pi}{\rho_<}\right)}I_{n+\frac{1}{2}}(\tfrac{1}{2}\rho_<)\right]\left[\sqrt{\left(\frac{\pi}{\rho_>}\right)}K_{n+\frac{1}{2}}(\tfrac{1}{2}\rho_>)\right]P_n(\cos\alpha_1),$$

where

$$\rho_< \equiv \min(\rho,\rho_1),\ \rho_> \equiv \max(\rho,\rho_1),\ P_n(\cos\alpha_1) = \frac{4\pi}{2n+1}\sum_{m=-n}^{n}Y_n^{m*}(\theta,\phi)Y_n^m(\theta_1,\phi_1),$$

α_1 being the angle between the directions of $\boldsymbol{\rho}$, $\boldsymbol{\rho}_1$, Y_n^m denoting spherical harmonics, $I_{n+\frac{1}{2}}$, $K_{n+\frac{1}{2}}$ modified Bessel functions and (ρ,θ,ϕ), (ρ_1,θ_1,ϕ_1) the coordinates of $\boldsymbol{\rho}$, $\boldsymbol{\rho}_1$, respectively. Making use of the expansions

$$e^{\frac{1}{2}\rho\cos\theta} = \sum_{0}^{\infty}(2n+1)\sqrt{(\pi/\rho)}I_{n+\frac{1}{2}}(\tfrac{1}{2}\rho)P_n(\cos\theta), \quad (56)$$

$$\cos\theta\, e^{\frac{1}{2}\rho\cos\theta} = \sum_0^\infty (2n+1)\surd(\pi/\rho)[I_{n+\frac{3}{2}}(\tfrac{1}{2}\rho) + (2n/\rho)I_{n+\frac{1}{2}}(\tfrac{1}{2}\rho)]P_n(\cos\theta), \tag{57}$$

results in $$f(\rho,\theta) = \sum_0^\infty f_n(\rho)P_n(\cos\theta),$$

where

$$\begin{aligned} f_n(\rho) = {} & (\rho^{-3}+\rho^{-2})e^{-\frac{1}{2}\rho}\,\delta_{0n} \\ & -(2n+1)\,e^{-\rho}\{[(n+1)\rho^{-3}+\tfrac{3}{2}\rho^{-2}]\surd(\pi/\rho)I_{n+\frac{1}{2}}(\tfrac{1}{2}\rho) + \tfrac{1}{2}\rho^{-2}\surd(\pi/\rho)I_{n+\frac{3}{2}}(\tfrac{1}{2}\rho)\}. \end{aligned} \tag{58}$$

Since $$q(\rho,\theta) = \int_0^\infty \int_0^\pi \int_0^{2\pi} G(\boldsymbol{\rho},\boldsymbol{\rho}_1)f(\rho_1,\theta_1)\rho_1^2\sin\theta_1,\mathrm{d}\phi_1\,\mathrm{d}\theta_1\,\mathrm{d}\rho_1,$$

it follows from the above that

$$\left.\begin{aligned} \mathscr{T}_1^B(\rho,\theta) &= e^{\frac{1}{2}\rho\cos\theta}\sum_0^\infty t_n(\rho)P_n(\cos\theta), \\ t_n(\rho) &= -\pi^{-1}\int_0^\infty f_n(\rho_1)\surd(\pi/\rho_<)\,I_{n+\frac{1}{2}}(\tfrac{1}{2}\rho_<)\surd(\pi/\rho_>)K_{n+\frac{1}{2}}(\tfrac{1}{2}\rho_>)\,\rho_1^2\,\mathrm{d}\rho_1. \end{aligned}\right\} \tag{59}$$

(Any additional complementary integral which vanishes at infinity will be singular as $\rho\to 0$ and hence unmatchable.)

From (59) one readily finds that, as $\rho\to\infty$,

$$t_0(\rho) = O\left(\frac{\log\rho}{\rho}e^{-\frac{1}{2}\rho}\right), \quad t_n(\rho) = O\left(\frac{e^{-\frac{1}{2}\rho}}{\rho}\right) \quad (n\geqslant 1).$$

However, it is shown in appendix B that this logarithmic behaviour in $t_0(\rho)$ is cancelled by the summation of the additional terms in (59) and that, in fact, in the wake,

$$\mathscr{T}_1^B \sim \rho^{-1}[\log 2 - \tfrac{1}{2} - \gamma - \log\eta^2 - E_1(\eta^2)]\,e^{-\eta^2} \quad (\eta \text{ fixed}, \rho\to\infty),$$

where $\eta \equiv \rho^{\frac{1}{2}}\sin\frac{1}{2}\theta$, $E_1(\zeta)$ is the exponential integral function $\left(\equiv\int_\zeta^\infty t^{-1}e^{-t}\,\mathrm{d}t\right)$ and γ is Euler's constant $(=0{\cdot}577\ldots)$.

On the other hand, as $\rho\to 0$, one has from (59) that

$$t_0(\rho) = O(1), \quad t_n(\rho) = O(\rho^{n-1}) \quad (n\geqslant 1).$$

Therefore, $$\mathscr{T}_1^B(\rho,\theta) \sim t_0(0) + t_1(0)\cos\theta + O(\rho) \quad (\rho\to 0). \tag{60}$$

Since $\cos\theta$ is not a harmonic function (and, therefore, not a complementary integral of the energy equation in the Stokes region) it follows that only $t_0(0)$ need be evaluated for matching purposes. From (59),

$$\begin{aligned} t_0(\rho) &= -\pi^{-1}\Big[\surd(\pi/\rho)K_{\frac{1}{2}}(\rho/2)\int_0^\rho f_0(\rho_1)\surd(\pi/\rho_1)\,I_{\frac{1}{2}}(\rho_1/2)\rho_1^2\,\mathrm{d}\rho_1 \\ &\qquad + \surd(\pi/\rho)\,I_{\frac{1}{2}}(\rho/2)\int_\rho^\infty f_0(\rho_1)\surd(\pi/\rho_1)K_{\frac{1}{2}}(\rho_1/2)\rho_1^2\,\mathrm{d}\rho_1\Big] \\ &= \rho^{-1}[(-\log\rho + 2\log 2 - \gamma - E_1(\rho))e^{-\rho/2} + 2(E_1(2\rho) - E_1(\rho))e^{\rho/2}]. \end{aligned} \tag{61}$$

Expanding (61) for small ρ, noting that

$$E_1(\zeta) \sim -\log\zeta - \gamma + \zeta + O(\zeta^2) \quad (\zeta \to 0),$$

results in

$$t_0(0) = 1 - 2\log 2.$$

2.4. *Thermal field* (*inner region*)

Representing the Stokes temperature distribution as

$$T^B(r,\theta;R;\epsilon) \sim \sum_{n=1} \phi_n(R;\epsilon) T^B_n(r,\theta) \quad (r \text{ fixed}, \epsilon = o(1), R \to 0),$$

it follows from (52) and (60) that $\phi_1(R;\epsilon) = \epsilon R$, the governing equation for T^B_1 being obtained from the terms of $O(\epsilon R)$ in (42):

$$\nabla^2 T^B_1 = (-r^{-2} + \tfrac{3}{2}r^{-3} - \tfrac{1}{2}r^{-5}) P_1(\cos\theta), \tag{62}$$

the inhomogeneous term arising from $(\mathbf{v}^B_1 . \nabla) T_0$. The general solution of (62) is

$$T^B_1(r,\theta) = (\tfrac{1}{2} - \tfrac{3}{4}r^{-1} - \tfrac{1}{8}r^{-3}) P_1(\cos\theta) + \sum_0^\infty (A_n r^n + B_n r^{-n-1}) P_n(\cos\theta). \tag{63}$$

Based upon (60), matching considerations indicate that

$$A_0 = t_0(0) = 1 - 2\log 2, \quad A_n = 0 \quad (n \geqslant 1),$$

the remaining constants being determined from the boundary condition $T^B_1(1,\theta) = 0$: $\quad B_0 = 2\log 2 - 1, \quad B_1 = \tfrac{3}{8}, \quad B_n = 0 \quad (n \geqslant 2).$

As in the forced convection problem, it is possible at this point to determine another term in the inner temperature expansion without having to obtain additional terms in the outer expansion. One readily finds that the governing equation for the term of $O(\epsilon R^2)$ in T^B involves inhomogeneous terms arising from $(\mathbf{v}^B_1 . \nabla) T^F_1$ and $(\mathbf{v}^F_0 . \nabla) T^B_1$, the latter giving rise to a particular integral which includes the function $-\log r$. In the matching region, such a term behaves as $\epsilon R^2 \log R$. However, it can be readily shown that $\mathscr{T}^B$ does not contain a term of $O(\epsilon R^2 \log R)$ since, if it did, the governing equation for such a term would be the homogeneous Oseen energy equation, any solution of which, if vanishing at infinity, would be unmatchable with the inner expansion. Hence, it is required that T^B contains a term of $O(\epsilon R^2 \log R)$ which, in the matching region, cancels the behaviour arising from the $-\log r$ in the term of $O(\epsilon R^2)$. Setting $\phi_2(R,\epsilon) = \epsilon R^2 \log R$, the governing equation for T^B_2 is merely Laplace's equation. With the boundary condition $T^B_2(1,\theta) = 0$ and the matching requirement $T^B_2(r,\theta) \sim -1$ as $r \to \infty$, one obtains

$$T^B_2(r,\theta) = -(1 - r^{-1}). \tag{64}$$

2.5. *Higher-order velocity field* (*outer region*)

Setting $\Delta_2(R;\epsilon) = \epsilon R$ and $\Delta_3(R;\epsilon) = \epsilon^2$, the resulting equations for $\mathbf{V}^B_2$ and $\mathbf{V}^B_3$ are, respectively,

$$\left(\frac{\partial}{\partial x} - \nabla^2\right) \mathbf{V}^B_2 + \nabla \mathscr{P}^B_2 = -(\mathbf{V}^B_1 . \nabla) \mathbf{V}^F_1 - (\mathbf{V}^F_1 . \nabla) \mathbf{V}^B_1 + \mathscr{T}^F_2 \mathbf{i}_x, \tag{65}$$

$$\left(\frac{\partial}{\partial x} - \nabla^2\right) \mathbf{V}^B_3 + \nabla \mathscr{P}^B_3 = -(\mathbf{V}^B_1 . \nabla) \mathbf{V}^B_1 + \mathscr{T}^B_1 \mathbf{i}_x. \tag{66}$$

Since $\mathscr{T}_1^F$ and $\mathscr{T}_1^B$ are expressible only in series form, $\mathbf{V}_2^B$ and $\mathbf{V}_3^B$ must be similarly expressed. For matching purposes, however, only the behaviour of these velocities as $\rho \to 0$ is required; this is determined in appendix C by means of the fundamental solution of the Oseen momentum equation (cf. Lagerstrom 1964), the integrals having the following behaviour:

$$\Psi_2^B(\rho,\theta) \underset{\rho\to 0}{\sim} -\tfrac{3}{4}\rho\sin^2\theta - \tfrac{3}{8}\rho^2\sin^2\theta\cos\theta + \tfrac{1}{2}\rho^2\sin^2\theta\left(\tfrac{3}{8} + 0{\cdot}21 + \sum_0^\infty a_n\right) + O(\rho^3), \tag{67}$$

$$\Psi_3^B(\rho,\theta) \underset{\rho\to 0}{\sim} \tfrac{1}{2}\rho^2\sin^2\theta\left(\tfrac{5}{2}\log 2 - 2 + \sum_0^\infty b_n\right) + O(\rho^3), \tag{68}$$

where the constants $0{\cdot}21$ and $\frac{5}{2}\log 2 - 2$ arise from the inertial forces on the right-hand side of (65) and (66), respectively, the series Σa_n arising from $\mathscr{T}_2^F$ and Σb_n from $\mathscr{T}_1^B$, the approximate values being (cf. appendix C):

$$\sum_0^\infty a_n \approx 0{\cdot}74, \quad \sum_0^\infty b_n \approx -0{\cdot}29.$$

2.6. *Higher-order velocity field* (*inner region*)

From (67) it follows immediately that (cf. equation (49))

$$\lambda_2 = \tfrac{3}{8} + 0{\cdot}21 + \Sigma a_n \approx 1{\cdot}33, \tag{69}$$

thus completing the determination of $\psi_2^B(r,\theta)$. The behaviour of Ψ_3^B in (68) indicates the presence of a term of $O(\epsilon^2)$ in the inner region. Setting $\delta_3(R;\epsilon) = \epsilon^2$, the governing equation for ψ_3^B is the homogeneous Stokes equation. From (68) and the symmetry and surface conditions, it follows that

$$\psi_3^B(r,\theta) = \lambda_3(\tfrac{1}{2}r^2 - \tfrac{3}{4}r + \tfrac{1}{4}r^{-1})\sin^2\theta, \tag{70}$$

where

$$\lambda_3 = \tfrac{5}{2}\log 2 - 2 + \Sigma b_n \approx -0{\cdot}56. \tag{71}$$

The pressure associated with (70) is readily found to be

$$p_3^B(r,\theta) = -\tfrac{3}{2}\lambda_3 r^{-2}P_1(\cos\theta). \tag{72}$$

In a manner analogous to that which led to $\phi_2 T_2^B$, it is found that ψ^B contains the term $\frac{27}{80}\epsilon R^2 r^2 \log r \sin^2\theta$ (arising from the non-linear effects between $\mathbf{v}_1^B$, $\mathbf{v}_1^F$ and between $\mathbf{v}_2^B$, $\mathbf{v}_0^F$) and, setting $\delta_4(R;\epsilon) = \epsilon R^2 \log R$, one finds:

$$\psi_4^B = \tfrac{27}{40}(\tfrac{1}{2}r^2 - \tfrac{3}{4}r - \tfrac{1}{4}r^{-1})\sin^2\theta, \tag{73}$$

$$p_4^B = -\tfrac{81}{80}r^{-2}P_1(\cos\theta). \tag{74}$$

2.7. *Drag and heat transfer results*

In terms of the drag coefficient, $C_D \equiv \text{drag}/(\pi a^2 \frac{1}{2}\rho_\infty U_\infty^2)$, one has that

$$C_D \sim C^F(R) + C^B(R;\epsilon) \quad (\epsilon = o(1),\ R \to 0),$$

where C^F is given by Proudman & Pearson (1957) and C^B is obtainable from the present analysis:

$$C^F = \frac{12}{R}[1 + \tfrac{3}{8}R + \tfrac{9}{40}R^2\log R + O(R^2)], \tag{75}$$

$$C^B = \frac{12}{R}[\epsilon + (\lambda_2 - \tfrac{2}{3})\epsilon R + \lambda_3\epsilon^2 + \tfrac{27}{40}\epsilon R^2 \log R + O(\epsilon R^2) + O(\epsilon^2 R) + O(\epsilon^3)], \quad (76)$$

where $\lambda_2 \approx 1{\cdot}33$, $\lambda_3 \approx -0{\cdot}56$ and $\epsilon \equiv G/R^2$ (G, R being based upon the sphere radius and the result in (76) being for unit Prandtl number). The results in (75), (76) are indicated in figure 1, the negative values of G corresponding to $t_w < t_\infty$.

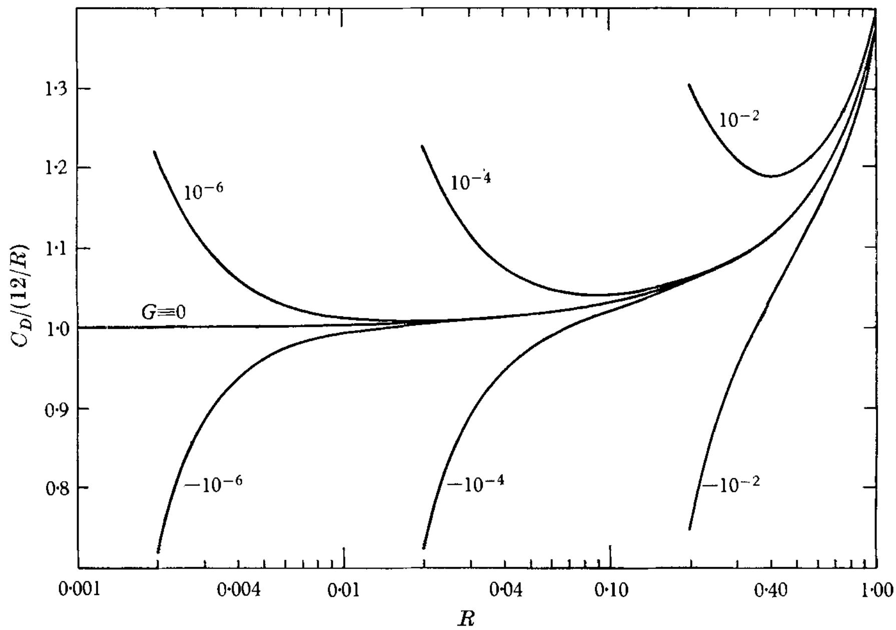

FIGURE 1. Drag coefficient for sphere in mixed convection flow (based upon equations (75) and (76)). Curves correspond to constant values of G.

Similarly, for the Nusselt number, $N \equiv Q/(4\pi ka(t_w - t_\infty))$, one has that

$$N \sim 1 + N^F(R) + N^B(R;\epsilon) \quad (\epsilon = o(1),\ R \to 0),$$

where the first term corresponds to pure thermal conduction, N^F is given by Rimmer (1968) and N^B is obtainable from the current theory (unit Prandtl number):

$$N^F = \tfrac{1}{2}R + \tfrac{1}{2}R^2 \log R + (\log 2 + \tfrac{1}{2}\gamma - \tfrac{133}{320})R^2 + \tfrac{1}{4}R^3 \log R + O(R^3), \quad (77)$$

$$N^B = (2\log 2 - 1)\epsilon R + \epsilon R^2 \log R + O(\epsilon R^2) + O(\epsilon^2 R), \quad (78)$$

where γ is Euler's constant, the coefficient of the $O(R^2)$ term in (77) is a corrected value and the term of $O(R^3 \log R)$ in (77), although not appearing in Rimmer's result, has been added on the basis of the analysis of Acrivos & Taylor (1962). One notes that N^B is second order. In addition, for $R \approx 0{\cdot}1$, the second term in (78) is comparable but of opposite sign to the first term, indicating that the applicability of (78) is for very small values of R.

2.8. *Some observations*

From the leading term in ψ^B, $\epsilon\psi_1^B(r,\theta)$, it is seen that, with respect to the inner region, the primary effect of buoyancy is to cause the apparent magnitude of the uniform stream to be $(1+\epsilon)U_\infty$. That is, the leading gravity effect in the inner region does not arise from the buoyant force in that region but rather is induced by the buoyancy effect of the outer region. As a result, the governing equation for ψ_1^B is the homogeneous Stokes equation, indicating that, even for the gravity-induced velocity field, diffusion is the predominant effect in the inner region.

As is implied in figure 1, the results of the present theory are also applicable if $t_w < t_\infty$ (G negative). In such a case, the buoyancy effect opposes the forced convection (as is seen in figure 1); for example, $\epsilon \mathbf{V}_1^B$ (ϵ negative) represents flow being drawn down axially through the wake and being pushed out radially in the irrotational region. For the case in which gravity acts in the same direction as the uniform stream, the present results are applicable provided ϵ is everywhere replaced by $-\epsilon$.

Consideration of global effects arising from (31) leads to some interesting results. Employing a spherical control volume (V_Γ) of radius Γ, the individual terms (based upon (29), (33), (40)) in the x-component of (31) can be readily integrated over V_Γ. In the limit of large Γ, such integrations indicate that the (x-component) momentum of the rotational flow in the wake is

$$4\pi\Gamma - 16\pi + O(1/\Gamma)$$

(times $\epsilon\rho_\infty \nu^2$), the $4\pi\Gamma$ being due to the buoyancy force and the -16π arising from equal contributions from the buoyancy force, the viscous shear in the wake, the pressure associated with the rotational velocity field and the viscous drag resulting from the irrotational velocity field (the x-component of $\mathbf{V}_{1i}^B$ being $1/\rho$). The fact that the last three effects are non-vanishing as $\Gamma \to \infty$ is indicative of a fundamental difference in character between natural and forced flow phenomena. In a word, as $\rho \to \infty$, the momentum effects associated with $\mathbf{V}^B$ exceed the corresponding effects of the forced flow field by $O(\rho)$. As a result, for large ρ, the disturbance velocity field is characterized by $\mathbf{V}^B$.

One notes that a closed-form solution for $\Psi_1^B(\rho,\theta)$ is peculiar to the case $\sigma = 1$. However, for $\sigma \neq 1$ the qualitative description should be the same as in the present case; that is, the gravity-induced velocity field is still of magnitude $[g\beta(t_w - t_\infty)a]/U_\infty$ with the wake region being characterized by a non-vanishing but bounded velocity and the flow in the inner region being diffusion-dominated. In fact, the behaviour of the outer velocity field in the matching region can be readily determined by employing the fundamental solution Γ_{ij} of the Oseen momentum equation (cf. appendix C). Integrating $\Gamma_{11}(\rho,\theta)\mathscr{T}_1^F(\rho,\theta)$ over physical space (noting that, for σ arbitrary, $\mathscr{T}_1^F = (1/\rho)\,e^{-\frac{1}{2}\sigma\rho(1-\cos\theta)}$) one readily finds that, for $\sigma \neq 1$, $\mathbf{V}_1^B$ behaves as a uniform stream of magnitude $\log\sigma/(\sigma-1)$ as $\rho \to 0$. Hence, defining $\gamma_1(\sigma)$ such that

$$\gamma_1(\sigma) = \begin{cases} \log\sigma/(\sigma-1), & \sigma \neq 1, \\ 1, & \sigma = 1, \end{cases}$$

it is noted that γ_1 is a continuous monotonically decreasing positive function ('decreasing' is a direct consequence of the fact that the thermal field contracts as σ increases) and that, for σ arbitrary,

$$\mathbf{V}^B \sim \gamma_1(\sigma)\epsilon\mathbf{i}_x, \quad \rho \to 0.$$

Finally, it should be reiterated that the results of the present analysis are based upon the assumptions that viscous dissipation is negligible, that the stratification of the undisturbed fluid can be neglected except with regard to the pressure gradient and that the fluid density is uniform except as it relates the density to the buoyancy effect (then taken to be only a function of temperature). For an ideal gas, these assumptions are equivalent to requiring that the parameters Δ, $\epsilon M^2/(\Delta R)^2$ and $M^2/(\Delta R)$ be arbitrarily small (where $\Delta \equiv (t_w - t_\infty)/t_\infty$ and $M \equiv$ Mach number).

3. Conjectures: the natural convection problem

Based upon the results of the preceding sections, it seems plausible to assume that $\mathbf{V}_1^*$ also behaves as a uniform stream in the matching region. That is, one expects that

$$\mathbf{V}_1^* \sim \beta_1(\sigma)\mathbf{i}_x + O(\rho^*) \quad (\rho^* \to 0), \tag{79}$$

$\beta_1(\sigma)$ being a constant (for given σ) whose evaluation is dependent upon the solving of (18)–(20). From (79) it then follows that the leading term in $\mathbf{v}^*(r,\theta;G)$ is $O(G^{\frac{1}{2}})$ rather than $O(G)$. Denoting this term by $G^{\frac{1}{2}}\mathbf{v}_0^*(r,\theta)$, one finds from (8) and (10) that

$$\nabla^2\mathbf{v}_0^* = \nabla p_0^*$$

and, therefore, employing (79) and the symmetry and surface boundary conditions, one obtains

$$\psi_0^*(r,\theta) = \beta_1(\sigma)(\tfrac{1}{2}r^2 - \tfrac{3}{4}r + \tfrac{1}{4}r^{-1})\sin^2\theta, \tag{80}$$

$$p_0^*(r,\theta) = -\tfrac{3}{2}\beta_1(\sigma)r^{-2}P_1(\cos\theta). \tag{81}$$

It immediately follows that the leading thermal convection effect in the inner region is $O(G^{\frac{1}{2}})$. Hence, setting $\phi_1^*(G) = G^{\frac{1}{2}}$, one obtains, in a manner analogous to that which led to T_1^B, that

$$T_1^*(r,\theta) = -\beta_2(\sigma)(1-r^{-1}) + \sigma\beta_1(\sigma)(\tfrac{1}{2} - \tfrac{3}{4}r^{-1} + \tfrac{3}{8}r^{-2} - \tfrac{1}{8}r^{-3})P_1(\cos\theta), \tag{82}$$

where $\beta_2(\sigma)$, a constant (for given σ), is defined in terms of the outer temperature field:

$$\mathscr{T}_1^*(\rho^*,\theta) \sim (1/\rho^*) - \beta_2(\sigma) + \tfrac{1}{2}\beta_1(\sigma)P_1(\cos\theta) + O(\rho^*) \quad (\rho^* \to 0). \tag{83}$$

Based upon (80) and (81) the drag coefficient, $C^* \equiv \text{drag}/(\pi a^2 \tfrac{1}{2}\rho_\infty U_B^2)$, is

$$C^* \sim \frac{12\beta_1(\sigma)}{G^{\frac{1}{2}}} \quad (G \to 0). \tag{84}$$

Similarly, from (82) one finds the Nusselt number to be:

$$N^* \sim 1 + \beta_2(\sigma)G^{\frac{1}{2}} \quad (G \to 0). \tag{85}$$

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Appendix A. Structure of expansions

$$\hat{\mathbf{v}} \sim U_\infty \begin{cases} \mathbf{i}_x + R\mathbf{V}_1^F + O(R^2) + \epsilon\mathbf{V}_1^B + \epsilon R\mathbf{V}_2^B + \epsilon^2\mathbf{V}_3^B + O(\epsilon R^2 \log R) & \text{outer}, \\ \mathbf{v}_1^F + O(R) + \epsilon\mathbf{v}_1^B + \epsilon R\mathbf{v}_2^B + \epsilon^2\mathbf{v}_3^B + O(\epsilon R^2 \log R) & \text{inner}, \end{cases}$$

$$t - t_\infty \sim (t_w - t_\infty) \begin{cases} R\mathscr{T}_1^F + R^2\mathscr{T}_2^F + O(R^3 \log R) + \epsilon R\mathscr{T}_1^B + O(\epsilon R^2 \log R) & \text{outer}, \\ T_0 + RT_1^F + O(R^2 \log R) + \epsilon RT_1^B + O(\epsilon R^2 \log R) & \text{inner}. \end{cases}$$

The causal relations between these terms is indicated below, the effects of buoyancy, convection and matching being denoted, respectively, by $\mathscr{B}$, $\mathscr{C}$, $\mathscr{M}$.

$$\begin{aligned}
&\mathscr{B}(\mathscr{T}_1^F) \to \mathbf{V}_1^B, \\
&\mathscr{M}(\mathbf{V}_1^B) \to \mathbf{v}_1^B, \\
&\mathscr{C}(\mathbf{V}_1^B, \mathscr{T}_1^F) \to \mathscr{T}_1^B, \\
&\mathscr{C}(\mathbf{v}_1^B, T_0) + \mathscr{M}(\mathscr{T}_1^B) \to T_1^B, \\
&\mathscr{B}(\mathscr{T}_2^F) + \mathscr{C}(\mathbf{V}_1^F, \mathbf{V}_1^B) + \mathscr{M}(\mathbf{v}_1^B) \to \mathbf{V}_2^B, \\
&\mathscr{B}(T_0) + \mathscr{C}(\mathbf{v}_0^F, \mathbf{v}_1^B) + \mathscr{M}(\mathbf{V}_2^B) \to \mathbf{v}_2^B, \\
&\mathscr{B}(\mathscr{T}_1^B) + \mathscr{C}(\mathbf{V}_1^B, \mathbf{V}_1^B) \to \mathbf{V}_3^B, \\
&\mathscr{M}(\mathbf{V}_3^B) \to \mathbf{v}_3^B.
\end{aligned}$$

Appendix B. Behaviour of $\mathscr{T}_1^B$ for large ρ

Examining the governing equation for $\mathscr{T}_1^B$, (53), one finds that integration of the inhomogeneous term over the control volume V_Γ results in the value $2\pi + O(\Gamma^{-1})$; a similar integration of the left-hand side results in (employing the divergence theorem, noting that $\mathscr{T}_1^B = O(1)$ as $\rho \to 0$)

$$\iint \frac{\partial \mathscr{T}_1^B}{\partial \rho}\, \mathrm{d}S_\Gamma - \iint \mathscr{T}_1^B \cos\theta\, \mathrm{d}S_\Gamma, \tag{B 1}$$

S_Γ being the surface of V_Γ. As $\Gamma \to \infty$, the only contribution in (B 1) is due to the wake region wherein $\partial/\partial\rho = O(\rho^{-1})$, $\cos\theta \sim 1 + O(\rho^{-1})$, indicating the first term in (B 1) is negligible in comparison with the second; hence one obtains the condition:

$$\iint \mathscr{T}_1^B\, \mathrm{d}S_{\text{wake}} = -2\pi, \tag{B 2}$$

S_{wake} being the surface area common to S_Γ and the wake. (Physically, (B 2) represents the fact that, in the wake, the thermal convection of $\mathscr{T}_1^B$ by the uniform stream is cancelled by that due to $(\mathbf{V}_1^B . \nabla)\mathscr{T}_1^F$ (the inhomogeneous term in (54)); this is necessary since these are the only two effects corresponding to a heat transfer rate of order $\epsilon ka(t_w - t_\infty)$, the heat transfer from the body being of order $\epsilon Rka(t_w - t_\infty)$). Were $\mathscr{T}_1^B$ of $O(\log\rho/\rho)$ in the wake, the left-hand side of (B 2) would be of $O(\log\Gamma)$; therefore, in order that (B 2) be satisfied it would be necessary that the θ-integration over S_{wake} of such a term be zero. However, as indicated in §2.3, $e^{\frac{1}{2}\rho\cos\theta}t_0(\rho)$ is the only term in the series expansion of $\mathscr{T}_1^B$, equation (59), which is $O(\log\rho/\rho)$ in the wake; one readily finds from (61) that the integration of $e^{\frac{1}{2}\rho\cos\theta}t_0(\rho)$ over S_{wake} results in the value $-4\pi\log\Gamma+O(1)$. On the basis of (B 2), then, it is concluded that the $O(\log\rho/\rho)$ behaviour of $e^{\frac{1}{2}\rho\cos\theta}t_0(\rho)$ in the wake is cancelled by the cumulative effect of the remaining terms, $n \geqslant 1$, in (59); that is,

$$\sum_1^\infty t_n(\rho)P_n(\cos\theta) \sim \frac{\log\rho}{\rho}\,e^{-\rho/2}+O\left(\frac{e^{-\rho/2}}{\rho}\right) \quad (\theta = O(\rho^{-\frac{1}{2}}),\ \rho\to\infty).$$

(Since, for $n \geqslant 1$, $t_n(\rho) = O(e^{-\frac{1}{2}\rho}/\rho)$ as $\rho\to\infty$, it follows that the series (59) is slowly convergent in the wake region.) As a result, in the wake, $\mathscr{T}_1^B = O(\rho^{-1})$. (This, in turn, assures that $\mathbf{V}_3^B$, the velocity induced by $\mathscr{T}_1^B$, is bounded.)

In a word, the series expansion (59) is a poor means of describing $\mathscr{T}_1^B$ in the wake region. A direct means of determining the latter behaviour is *via* the application of the limiting process: $\theta = O(\rho^{-\frac{1}{2}})$, $\rho\to\infty$ to the governing equation, (53). The resulting problem is made determinate by employing the global constraint, (B 2), and making use of the fact that, as $\rho\to\infty$, $\mathscr{T}_1^B$ is exponentially small except in the wake, wherein it is $O(\rho^{-1})$.

In applying a co-ordinate perturbation expansion to (53), it is convenient to follow the standard procedure of introducing cylindrical parabolic co-ordinates (ξ,η,ϕ) where $\xi \equiv \rho^{\frac{1}{2}}\cos\frac{1}{2}\theta$, $\eta \equiv \rho^{\frac{1}{2}}\sin\frac{1}{2}\theta$ and ϕ is the azimuthal angular co-ordinate. Therefore,

$$\frac{\partial}{\partial\rho} = \frac{1}{2}\frac{\xi}{\xi^2+\eta^2}\frac{\partial}{\partial\xi}+\frac{1}{2}\frac{\eta}{\xi^2+\eta^2}\frac{\partial}{\partial\eta}, \quad \frac{1}{\rho}\frac{\partial}{\partial\theta} = -\frac{1}{2}\frac{\eta}{\xi^2+\eta^2}\frac{\partial}{\partial\xi}+\frac{1}{2}\frac{\xi}{\xi^2+\eta^2}\frac{\partial}{\partial\eta}.$$

In the wake, corresponding to the limit: η fixed, $\xi\to\infty$, one then has

$$\frac{\partial}{\partial\rho} \sim \frac{1}{2\xi}\frac{\partial}{\partial\xi}+\frac{\eta}{2\xi^2}\frac{\partial}{\partial\eta} = O(\xi^{-2}), \quad \frac{1}{\rho}\frac{\partial}{\partial\theta} \sim \frac{1}{2\xi}\frac{\partial}{\partial\eta} = O(\xi^{-1})$$

('$\partial/\partial\xi = O(\xi^{-1})$' follows directly from the algebraic dependence of the flow quantities upon ξ, a basic characteristic of the wake). Letting $\mathscr{T}_1^B(\rho,\theta) \equiv H(\xi,\eta)$ and applying the above limiting process, equation (53) becomes ($\cos\theta \sim 1$, $\sin\theta \sim 2\eta/\xi$):

$$\left[\frac{1}{4\xi^2\eta}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial}{\partial\eta}\right)-\frac{1}{2\xi}\frac{\partial}{\partial\xi}+\frac{\eta}{2\xi^2}\frac{\partial}{\partial\eta}\right]H = \xi^{-4}(1-2\,e^{-\eta^2})\,e^{-\eta^2}.$$

Making use of the $O(\rho^{-1})$ behaviour of $\mathscr{T}_1^B$, one sets $H(\xi,\eta) = \xi^{-2}h(\eta)$, resulting in:

$$\left[\frac{d^2}{d\eta^2}+\left(2\eta+\frac{1}{\eta}\right)\frac{d}{d\eta}+4\right]h = 4(1-2\,e^{-\eta^2})\,e^{-\eta^2}.$$

Requiring that $h(\eta)$ be exponentially small as $\eta\to\infty$ and imposing condition (B 2) which, in current terms, is

$$\int_0^\infty h(\eta)\eta\,d\eta = -\tfrac{1}{4},$$

one finds that $$h(\eta) = [\log 2-\tfrac{1}{2}-\gamma-\log\eta^2-E_1(\eta^2)]e^{-\eta^2}, \tag{B 3}$$

the result being shown in figure 2.

Comparison with expansion (59) is instructive. Noting that $I_{n+\frac{1}{2}}$, $K_{n+\frac{1}{2}}$ are positive functions, it follows immediately from (58) and (59) that $t_n(\rho)$ is positive for $n \geqslant 1$. On the other hand, from (61) it is found that $t_0(\rho)$ is a negative function. Hence, it is seen from figure 2 that, towards the edge of the wake, (59) is predominated by $e^{\frac{1}{2}\rho\cos\theta}t_0(\rho)$ whereas, along the axis, the higher-order terms in (59) prevail. This is explainable by the fact that, for $\theta \neq 0$, the sign of $P_n(\cos\theta)$ varies with n and the terms in (59) corresponding to $n \geqslant 1$ tend to cancel one another; at $\theta = 0$, $P_n(\cos\theta) = 1$ for any n and the effect of the higher-order terms is additive. It is therefore concluded that, outside the wake (e.g. θ fixed $\neq 0$ and $\rho \to \infty$), $\mathscr{T}_1^B = O(\rho^{-1}\log\rho\, e^{-\frac{1}{2}\rho(1-\cos\theta)})$ whereas, within the wake, $\mathscr{T}_1^B = O(\rho^{-1})$.

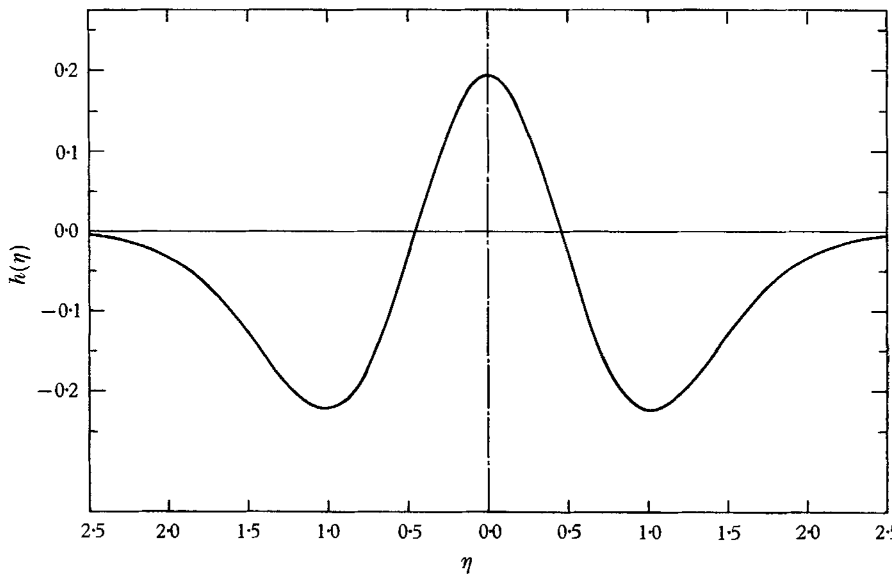

FIGURE 2. Behaviour of $\mathscr{T}_1^B$ in wake region (based upon equation (B 3)).

Appendix C. Higher-order velocity field

In dealing with the fundamental solution of the three-dimensional Oseen momentum equation, it is convenient to introduce Cartesian co-ordinates (x_1, x_2, x_3) with the x_1 and x_2 axes corresponding, respectively, to $\theta = 0$ and $\phi = 0$. Defining $\Gamma_{ij}(\mathbf{x})$ to be the i-component of the induced velocity at the origin arising from a unit force acting at $\mathbf{x}$ and directed in the j-direction, one has (Lagerstrom 1964):

$$\Gamma_{ij} = -\frac{1}{4\pi}\frac{\partial}{\partial x_i}\left[(1-e^{-\frac{1}{2}(\rho+x_1)})\frac{\partial}{\partial x_j}\log(\rho+x_1)\right] + \frac{e^{-\frac{1}{2}(\rho+x_1)}}{4\pi\rho}\,\delta_{ij}, \tag{C1}$$

(δ_{ij} being the Kronecker delta function). Therefore, if $\mathbf{F}(\mathbf{x})$ denotes the inhomogeneous term in the Oseen momentum equation, there exists a particular solution for the velocity having the following behaviour at the origin:

$$V_i(\mathbf{x}=0) = \iiint \Gamma_{ij}(\mathbf{x}')F_j(\mathbf{x}')\,d\tau', \tag{C2}$$

the integral being over all space.

Employing (C 1) and (C 2) and noting that the inhomogeneous terms in equations (65) and (66) are independent of ϕ, it is readily found that, for these cases, the induced velocity at the origin has zero components in the x_2 and x_3 directions.

Now, from (C 1):

$$\Gamma_{11} = \frac{1}{4\pi}\left[\frac{x_1}{\rho^3}+\left(-\frac{x_1}{\rho^3}+\frac{\frac{1}{2}(\rho-x_1)}{\rho^2}\right)e^{-\frac{1}{2}(\rho+x_1)}\right]$$

$$= \frac{1}{4\pi}\left[\frac{\cos\theta}{\rho^2}+\left(-\frac{\cos\theta}{\rho^2}+\frac{\frac{1}{2}(1-\cos\theta)}{\rho}\right)e^{-\frac{1}{2}\rho(1+\cos\theta)}\right], \quad \text{(C 3)}$$

$$\Gamma_{12} = \frac{x_2}{4\pi}\left[\frac{1}{\rho^3}-\left(\frac{1}{\rho^3}+\frac{\frac{1}{2}}{\rho^2}\right)e^{-\frac{1}{2}(\rho+x_1)}\right] = \Gamma(\rho,\theta)\cos\phi,$$

$$\Gamma_{13} = \frac{x_3}{4\pi}\left[\frac{1}{\rho^3}-\left(\frac{1}{\rho^3}+\frac{\frac{1}{2}}{\rho^2}\right)e^{-\frac{1}{2}(\rho+x_1)}\right] = \Gamma(\rho,\theta)\sin\phi,$$

where
$$\Gamma(\rho,\theta) \equiv \frac{\sin\theta}{4\pi}\left[\frac{1}{\rho^2}-\left(\frac{1}{\rho^2}+\frac{\frac{1}{2}}{\rho}\right)e^{-\frac{1}{2}\rho(1+\cos\theta)}\right]. \quad \text{(C 4)}$$

Hence, letting $F_{\tilde{\omega}}$ denote the component of $\mathbf{F}$ in the $\tilde{\omega}$ direction (cylindrical radial co-ordinate) and noting, as above, that $F_{\tilde{\omega}}$ is independent of ϕ, one has that

$$F_2 = F_{\tilde{\omega}}(\rho,\theta)\cos\phi, \quad F_3 = F_{\tilde{\omega}}(\rho,\theta)\sin\phi,$$

resulting in
$$\Gamma_{12}F_2+\Gamma_{13}F_3 = \Gamma(\rho,\theta)F_{\tilde{\omega}}(\rho,\theta).$$

Based on the above, it follows that there exist particular solutions (subscript 'p') of (65), (66) having the following behaviours at $\rho = 0$:

$$\mathbf{V}_{2p}^B\,(\rho = 0) = (\mathscr{I}_1+\mathscr{I}_2)\mathbf{i}_x, \quad \text{(C 5)}$$

$$\mathbf{V}_{3p}^B\,(\rho = 0) = (\mathscr{I}_3+\mathscr{I}_4)\mathbf{i}_x, \quad \text{(C 6)}$$

where

$$\mathscr{I}_1 \equiv \int_0^\infty\int_0^\pi\int_0^{2\pi}[-(\mathbf{V}_1^B.\nabla)u_1^F-(\mathbf{V}_1^F.\nabla)u_1^B+\mathscr{T}_2^F]\,\Gamma_{11}(\rho,\theta)\rho^2\sin\theta\,d\phi\,d\theta\,d\rho,$$

$$\mathscr{I}_2 \equiv \int_0^\infty\int_0^\pi\int_0^{2\pi}[-(\mathbf{V}_1^B.\nabla)w_1^F-(\mathbf{V}_1^F.\nabla)w_1^B]\,\Gamma(\rho,\theta)\rho^2\sin\theta\,d\phi\,d\theta\,d\rho,$$

$$\mathscr{I}_3 \equiv \int_0^\infty\int_0^\pi\int_0^{2\pi}[-(\mathbf{V}_1^B.\nabla)u_1^B+\mathscr{T}_1^B]\,\Gamma_{11}(\rho,\theta)\rho^2\sin\theta\,d\phi\,d\theta\,d\rho,$$

$$\mathscr{I}_4 \equiv \int_0^\infty\int_0^\pi\int_0^{2\pi}[-(\mathbf{V}_1^B.\nabla)w_1^B]\,\Gamma(\rho,\theta)\rho^2\sin\theta\,d\phi\,d\theta\,d\rho,$$

with $u_1^F \equiv \mathbf{V}_1^F.\mathbf{i}_x$, $w_1^F \equiv \mathbf{V}_1^F.\mathbf{i}_{\tilde{\omega}}$, etc. In evaluating $\mathscr{I}_1$ it is first necessary to determine $\mathscr{T}_2^F$. This is readily done by noting that the governing equation for $\mathscr{T}_2^F$ is of the same form as that for $\mathscr{T}_1^B$, equation (53), the inhomogeneous term in the former case arising from $(\mathbf{V}_1^F.\nabla)\mathscr{T}_1^F$; the boundary conditions are that $\mathscr{T}_2^F$ vanish as $\rho\to\infty$ and that, as $\rho\to 0$, $\mathscr{T}_2^F \sim \frac{1}{2}\rho^{-1}$ (in order to match the term $\frac{1}{2}r^{-1}$ in T_1^F). In a manner analogous to that leading to equation (59), one finds that

$$\mathscr{T}_2^F(\rho,\theta) = e^{\frac{1}{2}\rho\cos\theta}\sum_0^\infty q_n(\rho)P_n(\cos\theta),$$

where

$$q_n(\rho) = \tfrac{1}{2}\rho^{-1}\,e^{-\rho/2}\,\delta_{0n} - \frac{1}{\pi}\int_0^\infty s_n(\rho_1)\sqrt{(\pi/\rho_<)}\,I_{n+\frac{1}{2}}(\tfrac{1}{2}\rho_<)\sqrt{(\pi/\rho_>)}K_{n+\frac{1}{2}}(\tfrac{1}{2}\rho_>)\rho_1^2\,d\rho_1 \quad \text{(C 7)}$$

and

$$s_n(\rho) = -(\tfrac{3}{2}\rho^{-4}+\tfrac{3}{4}\rho^{-3})\,e^{-\frac{1}{2}\rho}\,\delta_{0n}+\tfrac{3}{4}\rho^{-3}\,e^{-\frac{1}{2}\rho}\delta_{1n} + (2n+1)(\tfrac{3}{2}\rho^{-4}+\tfrac{3}{2}\rho^{-3})\,e^{-\rho}\sqrt{(\pi/\rho)}I_{n+\frac{1}{2}}(\tfrac{1}{2}\rho).$$

One also notes that the evaluation of the integrals involving $\mathscr{T}_2^F$ and $\mathscr{T}_1^B$ necessitates employment of the following expansion (cf. equations (56) and (57)):

$$e^{\frac{1}{2}\rho\cos\theta}\Gamma_{11}(\rho,\theta) = (4\pi\rho^2)^{-1}\sum_0^\infty(2n+1)\Gamma_n(\rho)P_n(\cos\theta),$$

where

$$\Gamma_n(\rho) = \tfrac{1}{2}\rho e^{-\frac{1}{2}\rho}\,\delta_{0n}-\tfrac{1}{3}(1+\tfrac{1}{2}\rho)\,e^{-\frac{1}{2}\rho}\,\delta_{1n}+\sqrt{(\pi/\rho)}\,I_{n+\frac{3}{2}}(\tfrac{1}{2}\rho) + (2n/\rho)\sqrt{(\pi/\rho)}\,I_{n+\frac{1}{2}}(\tfrac{1}{2}\rho). \quad \text{(C 8)}$$

Lengthy computations result in:

$$\mathscr{I}_1 = \log 2-\tfrac{167}{480}+\sum_0^\infty a_n,$$

$$\mathscr{I}_2 = \tfrac{59}{15}\log 2-\tfrac{1979}{720}+C_1+C_2,$$

$$\mathscr{I}_3 = \tfrac{7}{6}\log 2-\tfrac{19}{18}+\sum_0^\infty b_n,$$

$$\mathscr{I}_4 = \tfrac{4}{3}\log 2-\tfrac{17}{18},$$

where

$$a_n \equiv \int_0^\infty q_n(\rho)\Gamma_n(\rho)\,d\rho, \quad \text{(C 9)}$$

$$C_1 \equiv \tfrac{3}{4}\pi\left[\frac{59}{768}+\frac{3\sqrt{2}}{128}-\frac{1}{16}\left(\frac{3-2\sqrt{2}}{3+2\sqrt{2}}\right)^{\frac{1}{2}}+\left(\frac{3-2\sqrt{2}}{3+2\sqrt{2}}\right)\left(\frac{\sqrt{2}}{96}-\frac{1}{16}\right) + \left(\frac{3-2\sqrt{2}}{3+2\sqrt{2}}\right)^{\frac{3}{2}}\left(\frac{161}{1920}+\frac{19\sqrt{2}}{320}\right)+\left(\frac{3-2\sqrt{2}}{3+2\sqrt{2}}\right)^{2}\left(\frac{19\sqrt{2}}{1920}+\frac{1}{80}\right)\right] \approx 0{\cdot}23,$$

$$\begin{aligned}C_2 \equiv \tfrac{3}{4}\pi\int_0^\infty [&-\tfrac{1}{2}\rho^{-4}+(\tfrac{3}{2}\rho^{-4}+\tfrac{3}{4}\rho^{-3}-\tfrac{1}{8}\rho^{-2})e^{-\frac{1}{2}\rho}I_0(\tfrac{1}{2}\rho)+(\rho^{-4}+\tfrac{1}{4}\rho^{-3}-\tfrac{1}{16}\rho^{-2})\,e^{-\frac{1}{2}\rho}I_1(\tfrac{1}{2}\rho)\\ &+(\tfrac{3}{2}\rho^{-4}+\tfrac{3}{4}\rho^{-3})\,e^{-\frac{1}{2}\rho}I_2(\tfrac{1}{2}\rho)+(-\tfrac{1}{2}\rho^{-4}-\tfrac{1}{2}\rho^{-3}-\tfrac{3}{32}\rho^{-2}-\tfrac{3}{32}\rho^{-1})\,e^{-\rho}I_0(\rho)\\ &+(-\rho^{-4}-\tfrac{3}{4}\rho^{-3}+\tfrac{1}{8}\rho^{-2})\,e^{-\rho}I_1(\rho)+(-\tfrac{1}{2}\rho^{-4}-\tfrac{1}{2}\rho^{-3})e^{-\rho}I_2(\rho)\\ &+(-\rho^{-4}-\rho^{-3}-\tfrac{1}{8}\rho^{-2}+\tfrac{1}{16}\rho^{-1})\,e^{-\rho}+(\tfrac{1}{2}\rho^{-4}+\tfrac{3}{4}\rho^{-3}+\tfrac{11}{32}\rho^{-2}+\tfrac{9}{64}\rho^{-1})\,e^{-\frac{3}{2}\rho}I_0(\tfrac{1}{2}\rho)\\ &+(\rho^{-4}+\tfrac{5}{4}\rho^{-3}+\tfrac{1}{4}\rho^{-2})\,e^{-\frac{3}{2}\rho}I_1(\tfrac{1}{2}\rho)+(\tfrac{1}{2}\rho^{-4}+\tfrac{3}{4}\rho^{-3})\,e^{-\frac{3}{2}\rho}I_2(\tfrac{1}{2}\rho)]\,d\rho\\ \approx -0{\cdot}34,&\end{aligned}$$

$$b_n \equiv \int_0^\infty t_n(\rho)\Gamma_n(\rho)\,d\rho, \quad \text{(C 10)}$$

$I_n(\rho)$ being the modified Bessel function of the first kind and $q_n(\rho)$ being defined in (C 7), $\Gamma_n(\rho)$ in (C 8) and $t_n(\rho)$ in (59). In determining C_2, recourse to a computer is necessitated, resulting in the indicated value. Finally, the values of a_n, b_n can be obtained in closed form, particular values being

$$a_0 = \tfrac{31}{32} \approx 0{\cdot}9688, \quad a_1 = -\tfrac{7}{32} \approx -0{\cdot}2188, \quad a_2 = \tfrac{131}{192} - \log 2 \approx -0{\cdot}0109,$$

$$b_0 = \tfrac{1}{2}\log 2 - 1 \approx -0{\cdot}6534, \quad b_1 = \tfrac{29}{40} - \tfrac{7}{10}\log 2 \approx 0{\cdot}2398,$$

$$b_2 = \tfrac{311}{168} - \tfrac{18}{7}\log 2 \approx 0{\cdot}0688.$$

Additional constants have been obtained (more readily) *via* quadrature:

$$a_3 \approx -0{\cdot}0016, \quad b_3 \approx 0{\cdot}0243, \quad b_4 \approx 0{\cdot}0111, \quad b_5 \approx 0{\cdot}0061.$$

These values are sufficient to indicate that

$$\sum_0^\infty a_n \approx 0{\cdot}74, \quad \sum_0^\infty b_n \approx -0{\cdot}29.$$

(The slow convergence of (59) in the wake region is apparently reflected in the series Σb_n: for any n, b_n arises directly from t_n *via* the buoyancy term in (66)).

Matching considerations indicate that $\Psi_2^B(\rho,\theta)$ must match the Stokeslet in $\psi_1^B(r,\theta)$. Hence, the required complementary integral in $\Psi_2^B(\rho,\theta)$ is

$$\Psi_{2c}^B(\rho,\theta) = -\tfrac{3}{2}(1+\cos\theta)(1-\mathrm{e}^{-\frac{1}{2}\rho(1-\cos\theta)}),$$

since, as $\rho \to 0$, one then has that

$$\Psi_{2c}^B(\rho,\theta) \sim -\tfrac{3}{4}\rho\sin^2\theta + \tfrac{3}{16}\rho^2\sin^2\theta(1-\cos\theta) + O(\rho^3). \tag{C 11}$$

In addition, the terms of $O(\rho^2)$ in $\Psi_2^B(\rho,\theta)$ must match the corresponding terms of $\psi_2^B(r,\theta)$. It is noted, however, that the coefficient of the $r^2\sin^2\theta\cos\theta$ term in $\psi_2^B(r,\theta)$, equation (49), is $-\frac{3}{8}$ whereas the coefficient of the corresponding term in (C 11) is $-\frac{3}{16}$. However, since the stream function $\rho^2\sin^2\theta\cos\theta$ corresponds to an x-component velocity of $\cos^3\theta + \cos\theta$, a term which is undefined at $\rho = 0$, it follows that the method employed in obtaining (C 5) is incapable of discerning the presence of such a term in $\mathbf{V}_2^B$. Therefore, the stream function corresponding to (C 5) is actually $\Psi_{2p}^B(\rho,\theta)$ where

$$\Psi_{2p}^B(\rho,\theta) \sim -\tfrac{3}{16}\rho^2\sin^2\theta\cos\theta + \tfrac{1}{2}\rho^2\sin^2\theta(\mathscr{I}_1+\mathscr{I}_2) + O(\rho^3) \quad (\rho \to 0).$$

No such complications arise in the case of $\mathbf{V}_3^B$, it being evident that Ψ_3^B must behave as a uniform stream as $\rho \to 0$ in order to match the term of $O(\epsilon^2)$ in the inner expansion, the governing equation for the latter being the homogeneous Stokes equation.

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